



Ideal operators and relative Godun sets

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Abstract: In this paper we study ideals in Banach spaces through ideal operators. We provide characterisation of recently introduced notion of *almost isometric* ideal which is a version of *Principle of Local Reflexivity* for a subspace of a Banach space. Studying ideals through ideal operators give us better insight in to the properties of these subspaces *vis-a-vis* properties of the space itself. We provide a few applications of our characterisation theorem.

Key words: Ideals, almost isometric ideals, strict ideals, maximal ideal operator, Godun sets, VN-subspaces.

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1. INTRODUCTION AND PRELIMINARIES

Principle of local reflexivity (henceforth PLR) states that finite dimensional subspaces of X^{**} , the bidual of a Banach space X , are almost isometric to finite dimensional subspaces of X . PLR also provides control over actions of a fixed finite dimensional subspace $F \subseteq X^*$ on a finite dimensional subspace $E \subseteq X^{**}$ and its almost isometric copy in X . It is immediately realised that PLR is a consequence of finite representability of X^{**} in X and X^* is norming for X^{**} , which in turn is a consequence of existence of norm one projection in X^{***} with range X^* and kernel X^\perp . Since finite representability is a notion which can be defined for arbitrary pair of Banach spaces, the notion of ideals were introduced and studied extensively. A closed subspace Y of a Banach space X is said to be an ideal in X if there exists a norm one projection $P : X^* \rightarrow X^*$ with $\ker(P) = Y^\perp$. The following characterisation of ideals, though it was widely known, was stated explicitly in [12].

*Professor Dutta passed away while the paper was being finalized.



THEOREM 1.1. *Let Y be a subspace of a Banach space X . Then Y is an ideal in X if and only if there exists a Hahn–Banach extension operator $\phi : Y^* \rightarrow X^*$ such that for every $\epsilon > 0$ and every finite dimensional subspaces $E \subset X$ and $F \subset Y^*$ there exists $T : E \rightarrow Y$ such that*

- (a) $Te = e$ for all $e \in Y \cap E$,
- (b) $\|Te\| \leq (1 + \epsilon)\|e\|$ for all $e \in E$,
- (c) $\phi f^*(e) = f^*(Te)$ for all $e \in E$, $f^* \in F$.

Clearly the notion of ideal imitates PLR. However it lacks two important aspects of X in X^{**} situation; namely, we do not get almost isometry from the finite dimensional subspace $E \subseteq X$ to Y and $\text{Range}(P)$ is not norming for X , where by norming we mean 1-norming. So following two possible strengthening of notion of ideals were considered.

DEFINITION 1.2. [9] A subspace Y of a Banach space X is said to be a strict ideal in X if Y is an ideal in X and $\text{Range}(P)$ is norming for X where $P : X^* \rightarrow X^*$ is a norm one projection with $\ker(P) = Y^\perp$.

DEFINITION 1.3. [2] A subspace Y of a Banach space X is said to be an almost isometric ideal (henceforth *ai*-ideal) in X if for every $\epsilon > 0$ and every finite-dimensional subspace $E \subset X$ there exists $T : E \rightarrow Y$ which satisfies condition (a) in Theorem 1.1 and $(1 - \epsilon)\|e\| \leq \|Te\| \leq (1 + \epsilon)\|e\|$ for all $e \in E$.

On the other hand there is a notion of u -ideal, which is a generalisation of M -ideal and is also a strengthening of PLR.

DEFINITION 1.4. [9] A subspace Y of a Banach space X is said to be a u -ideal in X if there exists a norm one projection $P : X^* \rightarrow X^*$ with $\ker(P) = Y^\perp$ and $\|I_{X^*} - 2P\| = 1$, where I_{X^*} denotes the identity operator on X^* .

It is straightforward to see that strict ideals are *ai*-ideals and *ai*-ideals are of course ideals. The inclusions are strict as shown by examples in [2].

If we view above notion of ideals and its subsequent strengthening as generalisations of X in X^{**} situation, many isometric properties of X are carried to Y and much of the studies in this area are devoted to that.

However PLR can be viewed simply as: identity operator on X^{**} is an extension of identity operator on X and X^* is norming for X . On the same

vein, the property of Y being an ideal in X may be viewed as there exists $T : X \rightarrow Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = I_Y$. To see this consider the following.

Suppose $Y \subseteq X$ and $T : X \rightarrow Y^{**}$ is such that $T|_Y = I_Y$ and $\|T\| \leq 1$. We will refer T as an ideal operator. Given T we define $P : X^* \rightarrow X^*$ as $P(x^*) = T^*|_{Y^*}(x^*|_Y)$ and $\phi : Y^* \rightarrow X^*$ by $\phi(y^*) = T^*|_{Y^*}(y^*)$. Then P is a norm one projection on X^* with $\ker(P) = Y^\perp$ and ϕ is a Hahn–Banach extension operator. In the sequel, we will refer these P and ϕ as ideal projection and Hahn–Banach extension operator corresponding to T . We also note that this definition is reversible, in the sense that given an ideal projection P , we may define T as above. Same is true for ϕ .

With this view point, in Section 2 of this paper we provide characterisations of ideal, strict ideal and *ai*-ideal in terms of the properties of ideal operator T . In the case of ideal and strict ideal (see Proposition 2.1) the results are essentially known.

Proposition 2.1 has some immediate corollaries. It is straightforward to see that if Y is 1-complemented in X then Y is an ideal in X (if $P : X \rightarrow X$ is a projection with $\|P\| = 1$ then P^* is an ideal projection). For a space Y which is 1-complemented in its bidual Y^{**} we show Y is an ideal in any superspace X if and only if Y is 1-complemented in X . In particular $L_1(\mu)$ or any reflexive space is an ideal in a superspace if and only if it is 1-complemented. It also follows that any ideal in $L_1(\mu)$ is 1-complemented.

Coming to *ai*-ideals we show that reflexive spaces with a smooth norm cannot have any proper *ai*-ideal. Situation becomes more interesting for the space $C[0, 1]$, the space of all real-valued continuous functions on $[0, 1]$ equipped with the supremum norm. It is known that $C[0, 1]$ is universal for the class of separable Banach spaces. We show that any *ai*-ideal of $C[0, 1]$ inherits the universality property from $C[0, 1]$. Also, any separable *ai*-ideal in $L_1(\mu)$, μ non-atomic, is isometric to $L_1[0, 1]$.

Section 3 of this paper is mainly devoted to the study of properties of ideal operator T . There is a need to exercise some caution while dealing with this operator. It may well be the case that for some ideal operator, Y has ‘nice’ property in X but there are other ideal operators for which such properties fail. For example, let Y be a *u*-ideal in Y^{**} . Then Y is always a strict ideal in Y^{**} under canonical projection π determined by $Y^{***} = Y^* \oplus Y^\perp$. But canonical projection may not satisfy *u*-ideal condition, namely, $\|I - 2\pi\| = 1$. So we introduce the notion of a maximal ideal operator (vis-a-vis, maximal ideal projection) and discuss properties of maximal ideal operator. However,

there are situations where there is only one possible ideal operator. Here we introduce relative Godun set $G(Y, X)$ of X with respect to Y . We recall that $Y \subseteq X$ is said to have *unique ideal property* (henceforth UIP) in X if there is only one possible norm one projection P on X^* with $\ker(P) = Y^\perp$. Similarly $Y \subseteq X$ is said to have *unique extension property* (henceforth UEP) in X if there is only one possible $T : X \rightarrow Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = I_Y$. From the above relation of P and T it is clear that Y has UIP in X if and only if it has UEP in X (and in this case ϕ is also unique).

If Y has UEP in Y^{**} then we just say Y has UEP.

UEP also provides a sufficient condition for an *ai*-ideal in a dual space to be a local dual for predual. See [6] and references there in for recent results on local duals. In particular a pertinent question in this area is if a separable Banach space with non separable dual always has a separable local dual.

We now provide some brief preliminaries needed throughout this paper.

To show that finite representability of X in a subspace Y with condition (a) of Theorem 1.1 is enough to characterise ideals through a global property we use following two lemmas from [12].

LEMMA 1.5. *Let E be a finite dimensional Banach space and $T : E \rightarrow Y^{**}$ be a linear map for any Banach space Y . Then there exists a net (T_α) , $T_\alpha : E \rightarrow Y$ such that*

- (a) $\|T_\alpha\| \rightarrow \|T\|$,
- (b) $T_\alpha e \rightarrow Te$ for all $e \in T^{-1}(Y)$,
- (c) $T_\alpha^* y^* \rightarrow T^* y^*$ for all $y^* \in Y^*$.

Next result from [12] shows that if we are given with T and (T_α) satisfying the conditions of Lemma 1.5, then we can modify (T_α) so that it satisfies the conditions of the following lemma.

LEMMA 1.6. *Let E be a finite dimensional Banach space and $T : E \rightarrow Y^{**}$ be a linear map for any Banach space Y . Let $F \subseteq Y^*$ be a finite dimensional Banach space, then there exists a net (T_α) , $T_\alpha : E \rightarrow Y$ such that*

- (a) $\|T_\alpha\| \rightarrow \|T\|$,
- (b) $T_\alpha e = Te$ for all $e \in T^{-1}(Y)$,
- (c) $T_\alpha^* y^* = T^* y^*$ for all $y^* \in F$,
- (d) $T_\alpha^* y^* \rightarrow T^* y^*$ for all $y^* \in Y^*$.

In Section 3 we will refer to VN-subspaces (very non-constraint subspaces) and their characterisation done in [3]. We recall definitions of VN-subspaces of a Banach space and nicely smooth spaces from [3] and [7] respectively.

DEFINITION 1.7. Let Y be a subspace of a Banach space X .

- (a) The ortho-complement $O(Y, X)$ of Y in X is defined as

$$O(Y, X) = \{x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y\}.$$

We denote $O(X, X^{**})$ by $O(X)$.

- (b) Y is said to be a VN-subspace of X if $O(Y, X) = \{0\}$.
(c) X is said to be nicely smooth if it is a VN-subspace of its bidual.

In this article, we consider only Banach spaces over the real field \mathbb{R} and all subspaces we consider are assumed to be closed.

2. GENERALISATION OF PLR

We start with the following proposition which is essentially known. However it provides a way to look ideals and strict ideals through some global property. Certain known properties of ideals and strict ideals follow trivially if we take this global view point.

PROPOSITION 2.1. *Let X be a Banach space and Y be a subspace of X . Then*

- (a) *Y is an ideal in X if and only if there exists $T : X \longrightarrow Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = I_Y$.*
(b) *Y is a strict ideal in X if and only if there exists an isometry $T : X \longrightarrow Y^{**}$ such that $T|_Y = I_Y$.*

We now note some immediate corollaries.

COROLLARY 2.2. *Let Y be a subspace of a Banach space X and Y be 1-complemented in its bidual. Then Y is an ideal in X if and only if Y is 1-complemented in X .*

Proof. If Y is 1-complemented in X then trivially Y is an ideal in X .

For the converse consider the map $T : X \longrightarrow Y^{**}$ from Proposition 2.1 (a). Let $P : Y^{**} \longrightarrow Y^{**}$ be a norm one projection with $\text{Range}(P) = Y$. If we take $Q = PT : X \longrightarrow X$ then $\|Q\| = \|PT\| \leq 1$, $Q^2 = Q$ and $\text{Range}(Q) = Y$. ■

COROLLARY 2.3. (a) If Y is isometric to any dual space then $Y \subseteq X$ is an ideal if and only if Y is 1-complemented in X . In particular no reflexive space can simultaneously be a VN-subspace and an ideal in any superspace.

(b) $L_1(\mu) \subseteq X$ is an ideal in X if and only if $L_1(\mu)$ is 1-complemented in X .

(c) Y is an ideal in $L_1(\mu)$ if and only if Y is 1-complemented in $L_1(\mu)$.

(d) No infinite dimensional reflexive space can be an ideal in a space with Dunford Pettis property (see [4, Definition 1.10]).

Proof. Proofs of (a) and (b) follow from Corollary 2.2.

(c) Let Y be an ideal in $L_1(\mu)$. Then Y^* is isometric to a 1-complemented subspace of $L_\infty(\mu)$. Thus Y is isometric to $L_1(\nu)$ for some positive measure ν . Hence Y is 1-complemented in its bidual. Then, by Corollary 2.2, Y is 1-complemented in $L_1(\mu)$.

(d) If Y is an infinite dimensional reflexive space and $Y \subseteq X$ is an ideal then by Corollary 2.2 Y is 1-complemented in X . However complemented subspaces of a space with Dunford Pettis property have Dunford Pettis property and a reflexive space with Dunford Pettis property is finite dimensional. ■

The notion of *ai*-ideals is strictly in between the notions of ideals and strict ideals. In the next theorem we characterise *ai*-ideals in terms of the operator T defined in Theorem 2.1.

DEFINITION 2.4. Let X and Z be Banach spaces. For $\epsilon > 0$, an operator $T : X \rightarrow Z$ is said to be an ϵ -isometry if $\|T\| \|T^{-1}\| \leq 1 + \epsilon$.

THEOREM 2.5. Let Y be a subspace of X . Then Y is an *ai*-ideal in X if and only if following condition is satisfied.

Given a finite dimensional subspace E of X and $\epsilon > 0$, there exists a bounded linear map $T^E : X \rightarrow Y^{**}$ such that $T^E|_Y = I_Y$ and T^E is an ϵ -isometry on $E \cap (T^E)^{-1}(Y)$.

Proof. Let E be a finite dimensional subspace of X . Without loss of generality, (by possibly adding an element from Y) we assume $E \cap Y \neq \{0\}$. Now consider a net (E_α) of finite dimensional subspaces of X such that $E_\alpha \supseteq E$ and the ϵ_{E_α} -isometry $\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}} : E_\alpha \rightarrow Y$ with $\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}}$ is identity on $E \cap Y$. Let T^E be a weak* limit point of this net in the sense that for all $x \in X$ and $x^* \in X^*$, $x^*(T^E x) = \lim_\alpha x^*(\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}} x)$. Let $T_\alpha = \widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}}|_E$ and $T_E = T^E|_E$.

It is straightforward to verify that

- (a) $\|T_\alpha\| \longrightarrow \|T_E\|$,
- (b) $T_\alpha y \longrightarrow T_E y$ for all $y \in T_E^{-1}(Y)$,
- (c) $T_\alpha^* y^* \longrightarrow T_E^* y^*$ for all $y^* \in Y^*$.

Thus T_α and the operator T_E satisfy conditions of Lemma 1.5.

Now applying a perturbation argument as in Lemma 1.6, given any $\epsilon > 0$ we can find S_α satisfying conditions of Lemma 1.6 and $\|S_\alpha - T_\alpha\| < \epsilon$ for large $\dim(E_\alpha)$. Hence S_α is $(\epsilon_{E_\alpha} + \epsilon)$ -isometry and $S_\alpha \longrightarrow T_E$ in the weak* topology. Thus T_E is an ϵ -isometry on $E \cap T^{-1}(Y)$.

Conversely, let E be a finite dimensional subspace of X and there exists an operator T^E satisfying the condition of the theorem. Consider $T^E|_E : E \rightarrow Y^{**}$ and apply PLR to get the desired operator satisfying the definition of *ai*-ideal. ■

Remark 2.6. Following example of *ai*-ideal in c_0 was considered in [2].

Let $Y = \{(a_n) \in c_0 : a_1 = 0\}$. Then $T : c_0 \longrightarrow Y^{**}$ in this case is given by $T(a) = (0, a_2, a_3, \dots)$. Hence T can not be extended as an isometry beyond $T^{-1}(Y)$.

COROLLARY 2.7. *Let $Y \subseteq X$ be an *ai*-ideal and Y be reflexive. If Y has UEP in X , then Y is isometric to X . In particular, in the following cases Y is isometric to X .*

- (a) *The norm of X is smooth on Y .*
- (b) *Y is a u -ideal in X [2, Theorem 2.3].*

Proof. Given $x \in X$, consider $E = \text{span}\{x\}$ and map $T^E : X \longrightarrow Y^{**}$ as in Theorem 2.5. Since Y is reflexive and $T^E|_Y = I|_Y$ we get T^E is onto. Thus $(T^E)^{-1}(Y) = X$. Since Y has UEP in X , there exists a unique T such that $T^E = T$ for all E . It follows that the ideal operator T is one-one as well. Hence T is an isometry. ■

Let μ be a non-atomic σ -finite measure. It is proved in [2] that any copy of ℓ_1 in $L_1(\mu)$ can not be an *ai*-ideal. However $L_1(\mu)$ contains a 1-complemented copy of ℓ_1 . Hence a copy of ℓ_1 can be an ideal in $L_1(\mu)$. It follows from Corollary 2.7 that any copy of ℓ_p in $L_p(\mu)$ can not be an *ai*-ideal for $1 < p < \infty$.

We will now prove that *ai*-ideals of $C[0, 1]$ are universal for separable Banach spaces.

PROPOSITION 2.8. *Let Y be an *ai*-ideal in $C[0, 1]$. Then Y is universal for separable Banach spaces.*

Proof. Let Y be an *ai*-ideal in $C[0, 1]$. Then Y is an L_1 -predual space and it follows from [2, Proposition 3.8] that Y inherits Daugavet property from $C[0, 1]$. Also from [13, Theorem 2.6] it follows that Y contains a copy of ℓ_1 . Thus Y is an L_1 -predual with non separable dual and hence Y is also universal for separable Banach spaces (see [10, Theorem 2.3]). ■

We now show that for a non-atomic measure μ , any separable *ai*-ideal in $L_1(\mu)$ is isometric to $L_1[0, 1]$.

PROPOSITION 2.9. *Let $Y \subseteq L_1(\mu)$ be a separable *ai*-ideal where μ is a non-atomic probability measure. Then Y is isometric to $L_1[0, 1]$.* ■

Proof. Let $Y \subseteq L_1(\mu)$ be a separable *ai*-ideal. Then by [2, Proposition 3.8], it follows that Y inherits Daugavet property from $L_1(\mu)$ and thus Y^* is non separable. Since Y is an ideal in $L_1(\mu)$ we have Y^* is isometric to a 1-complemented subspace of $L_\infty(\mu)$ and thus Y is isometric to $L_1(\nu)$ space for some measure ν . But Y has Daugavet property so ν can not have atoms. Now since ν is non atomic and Y is separable we can conclude that Y is isometric to $L_1[0, 1]$. ■

The property of being an *ai*-ideal is inherited from bidual.

PROPOSITION 2.10. *Suppose $Y \subseteq X$ and $Y^{\perp\perp}$ is an *ai*-ideal in X^{**} . Then Y is an *ai*-ideal in X^{**} and hence in particular in X .*

Proof. We note that the property of being *ai*-ideal is transitive. Since Y is always an *ai*-ideal in $Y^{\perp\perp}$ and $Y^{\perp\perp}$ is an *ai*-ideal in X^{**} , Y is *ai*-ideal in X^{**} as well. ■

We end this section stating a result which connects *ai*-ideals in a dual space to local dual of preduals.

DEFINITION 2.11. [6] A closed subspace Z of X^* is said to be a local dual space of a Banach space X if for every $\epsilon > 0$ and every pair of finite dimensional subspaces F of X^* and G of X , there exists an ϵ -isometry $L : F \rightarrow Z$ satisfying the following conditions.

- (a) $L(f)|_G = f|_G$ for all $f \in F$,
- (b) $L(f) = f$ for $f \in F \cap Z$.

The following result is simple to observe from above definition.

PROPOSITION 2.12. *Let X be a Banach space and Z be a subspace of X^* . If Z is a local dual of X , then Z is an ai -ideal in X^* .*

Remark 2.13. Let Y be an ai -ideal in a Banach space X . Since X is an ai -ideal in X^{**} , Y is an ai -ideal in X^{**} . But Y cannot be a local dual space of X^* as it is not norming for X^* . So the converse of Proposition 2.12 need not be true.

THEOREM 2.14. *Let X be a Banach space. Let Z be an ai -ideal in X^* with UEP and Z be norming for X . Then Z is a local dual space of X .*

Proof. Let F and G be finite dimensional subspaces of X^* and X respectively. Also, let $\epsilon > 0$. Now let $\hat{G} = \text{span}\{\hat{g}|_Z : g \in G\}$, where \hat{g} is the canonical image of g in X^{**} . Then, by [2, Theorem 1.4], there exists a Hahn–Banach extension operator $\varphi : Z^* \rightarrow X^{**}$ and an ϵ -isometry $L : F \rightarrow Z$ such that $Lf = f$ for all $f \in F \cap Z$ and $\varphi(\hat{g}|_Z)(f) = (\hat{g}|_Z)(Lf) = (Lf)(g)$ for all $g \in G$ and $f \in F$. Now to prove Z is a local dual space of X , it is enough to prove that $L(f)(g) = f(g)$ for all $f \in F$ and $g \in G$. Now let $g \in G$. Since Z is norming for X , \hat{g} is a Hahn–Banach extension of $\hat{g}|_Z$. Further, by UEP, \hat{g} is the only Hahn–Banach extension of $\hat{g}|_Z$. Therefore $(Lf)(g) = \varphi(\hat{g}|_Z)(f) = f(g)$ for all $f \in F$ and $g \in G$. Hence Z is a local dual space of X . ■

3. PROPERTIES OF IDEAL OPERATORS

In this section we explore conditions for the ideal operator T to be unique or one-one. Any nicely smooth space has UEP (see [3, 7]). However any Banach space X is a strict ideal in X^{**} . So for an ideal Y in X , to get uniqueness we mostly have to consider Y to be a strict ideal in X . As mentioned in the introduction, while considering UEP one needs to exercise some caution here. For an ideal Y in X we will first make sense of a maximal ideal projection through the use of Godun set of X with respect to Y .

We formulate the following lemma for which the equivalence of first three parts is established in [9, Lemma 2.2] and the proof for the fourth part goes verbatim as in the proof of $(2) \Rightarrow (4)$ and $(4) \Rightarrow (3)$ in [9, Proposition 2.3].

LEMMA 3.1. *Let Y be an ideal in X and T be the corresponding ideal operator. For $\lambda, a \in \mathbb{R}$, the following assertions are equivalent.*

- (a) $\|I - \lambda P\| \leq a$.
- (b) *For any $\epsilon > 0$, $x \in X$ and convex subset A of Y such that Tx is in the weak* closure of A there exists $y \in A$ such that $\|x - \lambda y\| < a\|x\| + \epsilon$.*
- (c) *For any $x \in X$ there exists a net $(y_\alpha) \subseteq Y$ such that (y_α) converges to Tx in the weak* topology and $\limsup_\alpha \|x - \lambda y_\alpha\| \leq a\|x\|$.*

Moreover, if Y is a strict ideal in X and T is the corresponding strict ideal operator then above assertions are also equivalent to the following.

- (d) *For $\epsilon > 0$ and any sequence (y_n) in B_Y with (y_n) converges in the weak* topology to Tx for some $x \in B_X$, there exist n and $u \in \text{co}\{y_k\}_{k=1}^n$, $t \in \text{co}\{y_k\}_{k=n+1}^\infty$ such that $\|t - \lambda u\| < a + \epsilon$.*

Let π be the canonical projection of X^{***} onto X^* . The Godun set $G(X)$ is defined to be $G(X) = \{\lambda : \|I - \lambda\pi\| = 1\}$ (see [9]).

For ideal $Y \subseteq X$ and ideal projection P we define Godun set of X with respect to Y and P as $G_P(Y, X) = \{\lambda : \|I - \lambda P\| = 1\}$. Then it follows that $0 \in G_P(Y, X)$ and $G_P(Y, X)$ is a closed convex subset of $[0, 2]$ and thus itself an interval.

Our next result is an analogue of [9, Lemma 2.5] and has interesting consequences.

LEMMA 3.2. *Let Y be a strict ideal in X and P be the corresponding strict ideal projection.*

- (a) *If Z is a subspace of X such that $Y \subseteq Z \subseteq X$ then there exists an ideal projection Q on Z^* with $\ker(Q) = Y^\perp$ such that $G_P(Y, X) \subseteq G_Q(Y, Z)$.*
- (b) *If Z is a closed subspace of Y then Y/Z is an ideal in X/Z and there exists an ideal projection \tilde{P} on $(X/Z)^*$ such that $G_P(Y, X) \subseteq G_{\tilde{P}}(Y/Z, X/Z)$.*

Proof. (a) Let T be the corresponding strict ideal operator from X to Y^{**} . Consider $T_Z = T|_Z : Z \rightarrow Y^{**}$. We define $Q : Z^* \rightarrow Z^*$ as $Q(z^*) = (T_Z)^*(z^*|_Y)$. It is straightforward to check that $\ker(Q) = Y^\perp \subseteq Z^*$. Thus Q is an ideal projection on Z^* . The proof now follows from equivalence of (d) and (a) in Lemma 3.1.

(b) We again let T be the corresponding strict ideal operator from X to Y^{**} . Let $q : X \rightarrow X/Z$ be the quotient map. We define $\tilde{T} : X/Z \rightarrow (Y/Z)^{**}$ by

$\tilde{T}(\bar{x}) = q^{**}(Tx)$, where \bar{x} is the equivalence class containing x . Let \tilde{P} be the ideal projection corresponding to \tilde{T} . Again a straightforward application of equivalence of (d) and (a) in Lemma 3.1 gives the desired conclusion. ■

For an ideal Y in X , we define Godun set of X with respect to Y as $G(Y, X) = \cup\{G_P(Y, X) : P \text{ is an ideal projection}\}$. We now verify that $G(Y, X)$ is $G_P(Y, X)$ for some ideal projection P . In the sequel we will refer such projection as maximal ideal projection and the corresponding T as maximal ideal operator.

THEOREM 3.3. *Let Y be an ideal in X . Then there exists an ideal projection P such that $G(Y, X) = G_P(Y, X)$.*

Proof. Suppose for all ideal projection P , $G_P(Y, X) = \{0\}$. Then $G(Y, X) = \{0\}$ and we choose any P as maximal ideal projection.

Suppose there exists an ideal projection P and $\lambda \in G_P(Y, X)$ with $\lambda \neq 0$. Then we claim that $[0, 1] \subseteq G_P(Y, X)$. To see this, suppose on the contrary that $G_P(Y, X) \subseteq [0, \gamma]$ for some $0 < \gamma < 1$. We choose $\mu \in (0, \gamma)$. It is straightforward to verify that $\gamma + \mu - \gamma\mu \in G_P(Y, X)$ as well. Thus $\gamma + \mu - \gamma\mu \leq \gamma$. Hence $\mu(1 - \gamma) \leq 0$ which is a contradiction.

Now let us consider $\lambda = \sup\{\mu : \mu \in G(Y, X)\}$.

If $\lambda \neq 0$, then by above argument either $\lambda = 1$ or $1 < \lambda \leq 2$. In the case $\lambda = 1$, there exists an ideal projection P such that $G(Y, X) = G_P(Y, X) = [0, 1]$.

If $\lambda > 1$, then choose a sequence (λ_n) in $G(Y, X)$ such that $\lambda_n > 1$ and λ_n converges to λ . Let P_n be an ideal projection corresponding to Y with $\|I - \lambda_n P_n\| = 1$ for all n . Since $B(X^*)$ is isometric to the dual of projective tensor product of X^* and X , there exists a bounded linear map $P : X^* \rightarrow X^*$ and a subsequence (denoted again by (P_n)) of (P_n) such that for every $x^* \in X^*$, $P_n(x^*)$ converges to $P(x^*)$ in the weak* topology. Since for every $x^* \in X^*$ and $n \in \mathbb{N}$, $P_n(x^*)$ is a Hahn-Banach extension of $x^*|_Y$, we can see that $P(x^*)$ is also a Hahn-Banach extension of $x^*|_Y$. Thus $\ker(P) = Y^\perp$. For any $x^* \in X^*$, since $x^* - P(x^*) \in Y^\perp = \ker(P)$, we can see that $P(P(x^*)) = P(x^*)$. Hence P is an ideal projection corresponding to Y . Since, for every $x^* \in X^*$, $x^* - \lambda_n P_n(x^*)$ converges to $x^* - \lambda P(x^*)$ in the weak* topology, we can see that $\|(I - \lambda P)(x^*)\| \leq \liminf_n \|(I - \lambda_n P_n)(x^*)\| \leq \|x^*\|$ for every $x^* \in X^*$. Thus $\|I - \lambda P\| = 1$. Hence $G(Y, X) = G_P(Y, X) = [0, \lambda]$. ■

Remark 3.4. We note that $G(Y, X) = \{0\}$ is possible. If $Y = \ell_1$ and

$X = Y^{**}$, then following the same argument used in [9, Proposition 2.6] it follows that $G(Y, X) = \{0\}$.

We now show that if Y is nicely smooth and Y embeds in a superspace X as a strict ideal then strict ideal operator T is unique.

PROPOSITION 3.5. *Let Y be a nicely smooth Banach space and Y be a strict ideal in a superspace X . Then the strict ideal operator is unique.*

Proof. Let T_1 and T_2 be two strict ideal operator. Then for any $x \in X$, $\|T_1x - y\| = \|T_2x - y\|$ for all $y \in Y$. Hence $T_1x - T_2x \in O(Y)$ (see [3]). But since Y is nicely smooth, $O(Y) = \{0\}$ and hence $T_1x = T_2x$. ■

We will now give a sufficient condition for a strict ideal Y in X to be a VN-subspace of X . We will first provide an analogue of [9, Proposition 2.7].

PROPOSITION 3.6. *Let Y be a strict ideal in X and P a strict ideal projection for Y in X^* . If $1 < \lambda \leq 2$ and $\|I - \lambda P\| = a < \lambda$ then for any proper subspace $M \subseteq X^*$, M norming for Y , we have M is weak* dense in X^* .*

Proof. Let $r_Y(M)$ be the greatest constant r such that $\sup_{x^* \in S_M} |x^*(y)| \geq r\|y\|$ for all $y \in Y$.

We will first show that for any weak* closed subspace $M \subseteq X^*$, $r_Y(M) \leq \lambda^{-1}a$. Without loss of generality let $M = \ker(x)$ for some $x \in S_X$.

Consider the isometry $T : X \rightarrow Y^{**}$ corresponding to P . Then by Lemma 3.1 there exists a net $\{y_\alpha\} \subseteq Y$ such that $y_\alpha \rightarrow Tx$ in the weak* topology and $\limsup \|x - \lambda y_\alpha\| \leq a\|x\|$.

Now since T is an isometry we have $\|y_\alpha\| \rightarrow 1$. For any $x^* \in S_M$, $\lambda|x^*(y_\alpha)| = |x^*(x - \lambda y_\alpha)| \leq \|x - \lambda y_\alpha\|$.

Since $\sup_{x^* \in S_M} |x^*(y_\alpha)| \geq r_Y(M)\|y_\alpha\|$ and $\|y_\alpha\| \rightarrow 1$ it follows that $r_Y(M) \leq \lambda^{-1}a$.

If there exists $1 < \lambda \leq 2$ and $\|I - \lambda P\| = a < \lambda$ then it follows that for any weak* closed proper subspace M of X^* which is norming for Y we have $r_Y(M) < 1$. This contradicts M is norming for Y and hence we have the result. ■

THEOREM 3.7. *Let $Y \subseteq X$ be a strict ideal such that $\|I - \lambda P\| < \lambda$ for some $1 < \lambda \leq 2$ where P is a strict ideal projection. Then Y is a VN-subspace of X . In particular a strict u -ideal is always a VN-subspace.*

Proof. Let $Y \subseteq X$ be a strict ideal such that $\|I - \lambda P\| < \lambda$ for some $1 < \lambda \leq 2$ where P is a strict ideal projection. Then it follows from Proposition 3.6 that any norming subspace for Y separates points in X . Hence Y is a VN-subspace of X (see [3]). ■

COROLLARY 3.8. *Let X be a Banach space. Then the following assertions are equivalent.*

- (a) X^* is separable.
- (b) There exists a renorming of X such that X is nicely smooth, that is X^* has no proper norming subspace.
- (c) There exists a renorming of X such that every subspace and quotient of X in the new norm are nicely smooth.

Proof. (a) \Rightarrow (b) From [9, Theorem 2.9] it follows that given $1 < \lambda < 2$ there exists a renorming of X such that $\lambda \in G(X)$. Conclusion follows from Theorem 3.7.

(b) \Rightarrow (c) Follows from Lemma 3.2 and Theorem 3.7.

(c) \Rightarrow (b) \Rightarrow (a) Is trivial. ■

COROLLARY 3.9. *Let Y be a strict ideal in X with strict ideal projection P and $O(Y, X) \neq \{0\}$. Then either $G_P(Y, X) = \{0\}$ or $G_P(Y, X) = [0, 1]$ and the later happens only if P is bicontractive.*

Proof. Let $0 \neq x \in O(Y, X)$ and $M = \ker(x)$. Then M is norming for Y . Now if we assume that $\|I - \lambda P\| < \lambda$ then $r_Y(M) \leq \|I - \lambda P\| \lambda^{-1} < 1$. But M is norming for Y so it follows that $\|I - \lambda P\| \geq \lambda$ and thus $G_P(Y, X) \subseteq [0, 1]$.

If $1 \notin G_P(Y, X)$ that is $\|I - P\| > 1$ then $G_P(Y, X) = \{0\}$. Hence the conclusion follows. ■

We now provide a sufficient condition for $x \in X$ to be in $O(Y, X)$.

PROPOSITION 3.10. *Let $Y \subseteq X$ be an ideal and T be the corresponding ideal operator. If $Tx = 0$, then $x \in O(Y, X)$. Consequently, if Y is also a VN-subspace of X , then any ideal operator T is one-one.*

Proof. Let $Tx = 0$. Then $Px^*(x) = 0$ for all $x^* \in X^*$ where P is the ideal projection corresponding to T . Thus $\text{Range}(P) \subseteq \ker(x)$. But $\text{Range}(P)$ is norming for Y , hence $\ker(x)$ is norming for Y and $x \in O(Y, X)$. ■

We now present an extension of [9, Theorem 7.4]. Towards this, for an ideal Y in X with associated ideal operator T , we define

$$Ba(Y, X) = \{x \in X : \text{there exists } \{y_n\} \subseteq Y \text{ such that } y_n \longrightarrow Tx \\ \text{in the weak}^* \text{ topology}\}$$

and

$$k_u^Y(x) = \inf \left\{ a : Tx = \sum y_n \text{ in weak}^* \text{ topology and for any } n, \right. \\ \left. \left\| \sum_{k=1}^n \theta_k y_k \right\| \leq a, \theta_k = \pm 1 \right\}.$$

It follows from [11] that if Y does not contain a copy of ℓ_1 then $Ba(Y, X) = X$. As considered in [9], we will say pair (Y, X) has property u if $k_u^Y(x) < \infty$ for all $x \in X$. In this case by closed graph theorem there exists a constant C such that $k_u^Y(x) \leq C\|Tx\|$ for all $x \in Ba(Y, X)$. We denote least constant C by $k_u^Y(X)$.

We will need the following lemma.

LEMMA 3.11. *Let Y be a strict ideal in X such that Y does not contain a copy of ℓ_1 and T be a strict ideal operator. Then Y is a u -ideal in X if and only if $k_u^Y(X) = 1$.*

Proof. If Y is a u -ideal in X then the result follows from [9, Lemma 3.4].

Conversely, by following the similar arguments as in [9, Lemma 5.3] that in this case $\|I - 2P\| \leq k_u^Y(X)$ where P is the ideal projection corresponding to the ideal operator T . Thus $\|I - 2P\| = 1$ and Y is a strict u -ideal in X . ■

THEOREM 3.12. *Let Y be a Banach space not containing ℓ_1 . Then the following assertions are equivalent.*

- (a) Y is a u -ideal in Y^{**} .
- (b) Whenever Y is a strict ideal in X , Y is a strict u -ideal in X .
- (c) Whenever Y is a strict ideal in X , $k_u^Y(X) < 2$.

Proof. (a) \Rightarrow (b) Since Y is a strict ideal in X , the ideal operator T is an extension of identity operator on Y and we have the results by [9, Proposition 3.6].

(b) \Rightarrow (c) Follows from Lemma 3.11.

(c) \Rightarrow (a) Follows from [9, Theorem 7.4] by taking $X = Y^{**}$. ■

We will now give examples where the ideal operator T is unique/ one-one.

We first see how does the ideal operator corresponding to an M -ideal in $C(K)$ behave. It is well-known that M -ideals in $C(K)$ are precisely of the form $J_D = \{f \in C(K) : f|_D = 0\}$ for some closed subset D of K .

PROPOSITION 3.13. *Let D be a closed subset of a compact Hausdorff space K . Then the following are equivalent.*

- (a) J_D is a strict ideal in $C(K)$.
- (b) J_D is a VN-subspace of $C(K)$.
- (c) $\overline{K \setminus D} = K$.

Proof. (a) \iff (b) Observe that J_D^* is norming if and only if $\overline{K \setminus D} = K$.
 (b) \iff (c) Follows from standard arguments. ■

EXAMPLE 3.14. Since the ideal projection corresponding to an M -ideal is unique, the ideal operator T corresponding to J_D is unique. In addition, if $\overline{K \setminus D} = K$, then, by Proposition 3.13, the unique ideal operator T corresponding to J_D is an isometry.

We next note that the ideal operators corresponding to $C(K, X)$ and $C(K, X^*)$ are isometries.

For a compact Hausdorff space K and for any Banach space X , let $WC(K, X)$ denote the space of X -valued functions on K that are continuous when X has the weak topology, equipped with the supremum norm. Also, $W^*C(K, X^*)$ denotes the space of X^* -valued functions on K that are continuous when X^* has the weak* topology, equipped with the supremum norm.

PROPOSITION 3.15. *Let X be a Banach space. Then $C(K, X)$ is a strict ideal in $WC(K, X)$. Moreover, $C(K, X^*)$ is a strict ideal in $W^*C(K, X^*)$.*

Proof. The former conclusion follows from the fact that there exists an isometry from $WC(K, X)$ to $C(K, X)^{**}$ whose restriction to $C(K, X)$ is the canonical embedding.

To prove the later conclusion recall that $C(K, X^*) = K(X, C(K))$, the space of compact operators from X to $C(K)$ and $W^*C(K, X^*) = L(X, C(K))$, the space of bounded linear operators from X to $C(K)$. It follows from [8, Lemma 2] that if Y is a Banach space having metric approximation property (in short MAP), then there exists an isometry from $L(X, Y)$ to $K(X, Y)^{**}$

whose restriction to $K(X, Y)$ is the canonical embedding. Since $C(K)$ has MAP, it follows that $C(K, X^*)$ is a strict ideal in $W^*C(K, X^*)$. ■

We know that $C(K, X) \subseteq Ba(K, X) \subseteq C(K, X)^{**}$, where $Ba(K, X)$ denotes the class of Baire-1 functions from K to X . Since $C(K, X)$ is a strict ideal in $C(K, X)^{**}$, it follows that $C(K, X)$ is also a strict ideal in $Ba(K, X)$. So the corresponding ideal operator is an isometry.

If X has MAP, then, by [8, Lemma 2], $K(X)$ is a strict ideal in $L(X)$. Now it follows from [5] that a reflexive space with compact approximation property has MAP. Hence if X is a reflexive space with compact approximation property such that either weak* denting points of B_{X^*} separates points of X^{**} or denting points of B_X separates points of X^* , then $K(X)$ is a strict ideal in $L(X)$ and is also a VN-subspace of $L(X)$ (see [3]). So the ideal operator T is an isometry.

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